Multidimensional ellipsoidally and spherically symmetric recursive digital-filter design with reduced number of optimised coefficients

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Abstract: In the paper, a method for the design of multidimensional ellipsoidally and spherically symmetric recursive digital filters using one-dimensional analogue filters is presented. The number of coefficients required to be optimised is very much reduced and the stability of the resultant filter can be guaranteed. This method is suitable for the design of moderately precise digital filters.

1 Introduction

Over the past decade, many methods [1-12] have been advanced for the design of two-dimensional circularly symmetric recursive digital filters. Generally, these methods utilised one of the following two techniques to obtain a circularly symmetric magnitude response:

(a) Two-dimensional frequency transformation from one-dimensional transfer function and a cascade of identical filters with different directions of recursion [1-6]
(b) Coefficient optimisation of transfer function possessing octagonal symmetry [7-12].

In general, the first technique has the advantage of ease in finding the filter coefficients. However, it can hardly be used for real-time applications due to realisation of additional cascade filters. The second technique is usually achieved by using a nonlinear optimisation. It could possibly provide an optimal or suboptimal solution. In general, the objective function of a recursive digital filter is nonlinear. Therefore, nonlinear optimisation is commonly used. Owing to the octagonal symmetric properties [13-14], the number of variables can be greatly reduced. Moreover, the denominator of the transfer function of such a two-dimensional filter is separable, i.e. a product of two one-dimensional polynomials of \( z_1 \) and \( z_2 \). With the help of the pole-inversion theorem [15], the stability of the transfer function can always be guaranteed.

Several papers have presented methods for designing FIR and IIR two-dimensional elliptical magnitude response digital filters [4, 16]. This kind of filter is useful in many situations, such as in the case of using different sampling rates in the two spatial directions due to compelling reason. This paper introduces a simple method on the design of two-dimensional elliptical and circular magnitude response recursive filters making use of one-dimensional analogue filters. The desired response is approximated by an additional zero-phase polynomial term in the numerator. One special feature of this method is that the employed one-dimensional analogue filters can be Butterworth, Chebyshev and elliptic filters, which are the major types of analogue filters and can be conveniently obtained from a filter handbook such as Reference 17. Hence, the number of coefficients required to be optimised can be kept to be minimum. Since circular magnitude response is a special case of the elliptical one, this method is also applicable to two-dimensional circular symmetric filter design.

2 One-dimensional filter

In this paper, we are concerned with \( M \)-dimensional digital filters. An ideal \( M \)-dimensional digital filter is a filter in which

\[
[H_d(z_1, \ldots, z_M)] = \begin{cases} 1, & \text{passband} \end{cases}
\]

\[
[0, & \text{stopband}]
\]

where \( z_i = \exp(jw_i), i = 1, 2, \ldots, M \).

Consider a one-dimensional analogue filter possessing a format as

\[
T(s) = A(s) = \frac{\prod_{k=1}^{P} (s - \omega_k)^{2m_k}}{\prod_{k=1}^{Q} (s - \omega_k)^{4m_k}}
\]

where \( K, P, \) and \( Q \) are positive integers and \( 2P < 2Q \). Applying the bilinear transformation [18],

\[
s = c \left( \frac{1 - z^{-1}}{1 + z^{-1}} \right)
\]

where \( c \) is determined by the one-to-one correspondence of a critical analogue frequency \( \omega_c \), and the corresponding critical digital frequency \( w_c \). In most cases, the critical frequency is the cutoff frequency. Hence, eqn. 2 becomes

\[
H(z) = \frac{\prod_{k=1}^{K} \tilde{D}_k(z)}{\prod_{k=1}^{Q} \tilde{D}_k(z)}
\]
where \( N_i(z) \) and \( D_i(z) \) are the numerator and denominator polynomials, which can be in the forms of

\[
\begin{align*}
N_i(z) &= \sum_{k=0}^{\infty} a_{ik} z^{-k} (1 + z^{-1})^{-2k} (1 - z^{-1})^{2k} \\
D_i(z) &= \sum_{k=0}^{\infty} b_{ik} z^{-k} (1 + z^{-1})^{-2k} (1 - z^{-1})^{2k}
\end{align*}
\]

(5)

3 Elliptical magnitude response two-dimensional digital filter

Supposing that

\[
H_j(z) = A_k \cdot N_j(z) \sum_{r=1}^{K_{(1)}} N_{(r)}(z) + z_1 \cdot z_2^{-1} \sum_{n=1}^{R} \sum_{m=1}^{S} c_{mn} \sin^2\left(\frac{2\pi w_1}{2}\right) \sin^2\left(\frac{2\pi w_2}{2}\right)
\]

possesses magnitude responses approximately equal to the desired magnitude responses along the \( w_1 \) and \( w_2 \) axes, i.e. the major and minor axes or vice versa. A two-dimensional elliptically symmetric digital filter can be formed as

\[
H_j(z_1, z_2) = A_k \cdot \prod_{k=1}^{K_{(1)}} N_j(z_1) \prod_{j=1}^{K_{(2)}} N_j(z_2)
\]

where \( z_1 = \exp(jw_1) \) and \( z_2 = \exp(jw_2) \). \( R \) and \( S \) are positive integers, and \( c_{mn} \), \( m = 1, \ldots, R \) and \( n = 1, \ldots, S \) are unknown parameters to be determined. For easy presentation, eqn. 7 has been expressed in terms of \( \sin^2(w_1/2) \) and \( \sin^2(w_2/2) \), which should be converted to \( z_1 \) and \( z_2 \) by

\[
\sin^2(w_i/2) = \frac{1}{2} \left( 1 - \frac{z_i + z_i^{-1}}{2} \right), \quad i = 1, 2
\]

(8)

Eqn. 7 consists of the sum of two parts. The first part is obtained by direct multiplication of eqns. 6a and 6b. It is known that the direct product of the 1-D digital filters will give a 2-D digital filter of rectangular (or square for the circularly symmetric case) contour in magnitude responses. To achieve elliptic (or circular for the circularly symmetric case) contour in magnitude responses, a correction part is needed. The second part (which is the correction part) is obtained by having a zero-phase polynomial consisting of the product of \( \sin^2(w_1/2) \) and \( \sin^2(w_2/2) \) in the numerator and having the same denominator as the first part. Hence, the degree of ellipticity (or circularity) of the magnitude responses can be controlled by the numerator coefficients \( c_{mn} \) of the second (or correction) part of eqn. 7.

Based on the above discussions, several points are noted as follows:

(a) The formats of the three major types of analogue filters, namely, Butterworth, Chebyshev and elliptic filters, can be expressed in a general form [19] as

\[
T_d(s) = A \frac{1}{s^{n} + b_{1}s^{n-1} + \cdots + b_{n}}
\]

for odd order and

\[
T_d(s) = A \prod_{k=1}^{n} \frac{a_{k}s^{2} + a_{0}}{s^{2} + b_{1}s + b_{0}}
\]

(9,10)

for even order, where \( a_{k} = 0 \) or 1. The filters in eqns. 9 and 10 are actually special cases of the filter shown in eqn. 2.

(b) The magnitude responses on the \( w_1 \) and \( w_2 \) axes of the transfer functions can be obtained easily by putting \( w_2 = 0 \) and \( w_1 = 0 \), respectively. It can be easily shown that

\[
|H_j(\exp(jw_1), \exp(0))| = |H_j(\exp(jw_2), \exp(0))|
\]

(11)

and

\[
|H_j(\exp(0), \exp(jw_2))| = |H_j(\exp(0), \exp(jw_1))|
\]

(12)

Therefore, the magnitude responses on the axes preserve the characteristics of \( H_j(z) \) and \( H_j(z) \).

(c) As the numerator of eqn. 7 is a zero-phase polynomial and the denominator is separable, eqn. 7 is quadrantly symmetric for 2-D elliptically symmetric magnitude response digital filters, and is octagonally symmetric for 2-D circularly symmetric digital filters.

(d) The denominator of eqn. 7 is separable, which is a product of the two one-dimensional filters. According to the property of the bilinear transformation, provided that the original analogue filter is stable, the stability of the transformed digital filter can be guaranteed. Hence, the filter expressed in the form of eqn. 7 must also be stable.

4 Circular magnitude response two-dimensional digital filter

It is obvious that the elliptically symmetric magnitude response case described in Section 3 is a generalised case of the circularly symmetric magnitude response one. Therefore, the principle can be applied to the design of circularly symmetric two-dimensional digital filters by adding an additional symmetric constraint,

\[
|H_j(\exp(jw_1), \exp(jw_2))| = |H_j(\exp(jw_2), \exp(jw_1))|
\]

(13)

which implies \( H_j(z) = H_j(z) \), \( R = S \) and \( c_{mn} = c_{nm} \). Hence, the number of independent coefficients and the number of frequency sample points required for optimisation can both be reduced by half.

5 Statement of the problem

Let \( C \) be the vector of the unknown coefficients

\[
C = [c_{mn}] \left[ m = 1, 2, \ldots, R \right] \left[ n = 1, 2, \ldots, S \right]^{T}
\]

(14)

and \( r \) represents the operation of matrix transposition. A performance error index \( E \) is defined as the sum of the squares of the errors of the \( M \) samples \( (w_{1p}, w_{2q}) \), \( p = 1, 2, \ldots, M \).

\[
E = \sum_{p=1}^{M} \left[ H_j(\exp(jw_{1p}), \exp(jw_{2q})) \right]^{2} - \left[ H_j(\exp(jw_{1q}), \exp(jw_{2q})) \right]^{2}
\]

(15)
where $H_d\exp(j\omega_{1d} \exp(j\omega_{2d}))$ represents the ideal magnitude responses of the digital filter. This statement of the problem is to find the optimal vector $\mathbf{C}^*$ which gives the minimum $E$. Such a $l_2$ norm optimisation can be carried out directly by using the unconstrained Fletcher-Powell algorithm [20].

6 Design procedures

The design procedures of the present method are listed as follows:

(a) Design two suitable one-dimensional analogue filters in the form of eqn. 2 to approximate the magnitude responses along the $w_1$ and $w_2$ axes. For the circular case, one one-dimensional analogue filter is enough. The analogue filters can be Butterworth, Chebyshev or elliptic analogue filters which can be obtained easily by looking up from an analogue-filter design table [17].

(b) Bilinear transform the analogue filters into digital filters as eqns. 4 and 5.

(c) Based on the transformed digital filters, form a new two-dimensional filter as eqn. 7.

(d) For the elliptical case, owing to the property of quadrantal symmetry, frequency sample points are chosen in the first quadrant of the $(w_1, w_2)$ plane. For the circular case, owing to the property of octagonal symmetry, frequency sample points can be chosen just in the $[0, 4\pi]$ sector of the $(w_1, w_2)$ plane.

(e) Find the optimal vector $\mathbf{C}^*$ which gives minimal $E$, say $E^*$, by an unconstrained optimisation algorithm.

7 Multidimensional digital filters

Eqn. 7 can be further extended for the $M$-dimensional ellipsoidal and spherical filter design. The following two equations can be used for the design of $M$-dimensional ellipsoidal magnitude response digital filters (see eqn. 16) and three-dimensional spherical magnitude response digital filters (see eqn. 17), respectively.

\[ H_d(z_1, z_2, \ldots, z_M) = A_1 \cdot A_2 \cdot A_3 \]

\[ \prod_{k_1=1}^{K_1} N_{d_1}(z_1) \cdot \prod_{k_2=1}^{K_2} N_{d_2}(z_2) + z_1^{K_1} \prod_{k_3=1}^{K_3} N_{d_3}(z_3) + z_2^{K_2} \prod_{k_4=1}^{K_4} N_{d_4}(z_4) + \sum_{k=1}^{K} D_{d_k}(z_1) \cdot D_{d_k}(z_2) \cdot D_{d_k}(z_3) \]

\[ H_d(z_1, z_2, z_3) = A_1^2 \]

\[ \prod_{k_1=1}^{K_1} N_{d_1}(z_1) \cdot \prod_{k_2=1}^{K_2} N_{d_2}(z_2) + z_1^{K_1} \prod_{k_3=1}^{K_3} N_{d_3}(z_3) + z_2^{K_2} \prod_{k_4=1}^{K_4} N_{d_4}(z_4) + \sum_{k=1}^{K} D_{d_k}(z_1) \cdot D_{d_k}(z_2) \cdot D_{d_k}(z_3) \]

where \( c_{\text{max}} = c_{\text{max}} = c_{\text{max}} = c_{\text{max}} = c_{\text{max}} \).

8 Design examples

Three design examples are shown below to illustrate the design method. They are, respectively, designs of two-dimensional elliptical magnitude response lowpass digital filter, two-dimensional circular magnitude response lowpass digital filter, and three-dimensional spherical magnitude response lowpass digital filter. The initial values of all the unknown coefficients are set to 1.

8.1 Example 1

A two-dimensional elliptical magnitude response digital filter with the following specifications:

Passband: in the $(w_1, w_2)$ plane, region inside the ellipse $(w_1/0.4\pi)^2 + (w_2/0.2\pi)^2 = 1$

Stopband: in the $(w_1, w_2)$ plane, region outside the ellipse $(w_1/0.65\pi)^2 + (w_2/0.4\pi)^2 = 0$

is designed with the use of a fourth-order one-dimensional Butterworth analogue filter as

\[ T(s) = \frac{1}{(s^2 + 1.8478s + 1)(s^2 + 0.7654s + 1)} \]

The corresponding one-dimensional digital filters for $w_1$ and $w_2$ axes are, respectively,

\[ H_d(z) = 0.004824 \frac{(1 + 2z^{-1} + z^{-2})(1 + 2z^{-1} + z^{-2})}{(1 - 0.3290z^{-1} + 0.0645z^{-2})(1 - 0.4531z^{-1} + 0.4663z^{-2})} \]

and

\[ H_d(z) = 0.004824 \frac{(1 + 2z^{-1} + z^{-2})(1 + 2z^{-1} + z^{-2})}{(1 - 0.6537z^{-1} + 0.2961z^{-2})(1 - 1.3210z^{-1} + 0.6328z^{-2})} \]

Fig. 1 Magnitude response of the filter in Example 1

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It is chosen that \( R = S = 2 \). A rectangular grid of 441 samples in the first quadrant is chosen for optimisation. The resultant filter obtained is

\[
0.02211 \cdot [1z_1^{-1}z_1^2z_1^3z_1^4] \times \left( \begin{array}{rrrrr}
1.0000 & -0.6232 & -2.6453 & 2.8349 & -0.7291 \\
-1.3179 & 0.5253 & 3.1449 & -3.8954 & 0.8925 \\
-0.9087 & 0.3664 & 1.4993 & -2.414 & 0.4816 \\
1.4457 & -1.1169 & 3.8036 & -0.7291 & -1.0499 \\
-0.3817 & 0.1979 & 1.4993 & -2.414 & 0.4816 \\
\end{array} \right] \times (1 - 0.3290z_1^{-1} + 0.06459z_1^2)(1 - 0.4531z_1^{-1} + 0.4663z_1^2) \times (1 - 0.9537z_1^{-1} + 0.29610z_1^2)(1 - 1.3210z_1^{-1} + 0.6328z_1^2)
\]

The optimal error index \( E^* = 5.7932 \). The maximum errors in passband and stopband are 0.053 and 0.034, respectively. Fig. 1 shows the magnitude response of the filter.

### 8.2 Example 2

A two-dimensional lowpass circular magnitude response digital filter with the passband and stopband radii as 0.25\( \pi \) and 0.5\( \pi \), respectively, is designed from a fourth-order one-dimensional Chebyshev analogue filter

\[
T(s) = \frac{1}{(s^2 + 1.179s + 0.5125)(s^2 + 0.4630s + 1.2196)}
\]

It is chosen that \( R = S = 2 \). A rectangular grid of 231 samples is chosen for optimisation. The resultant filter obtained is

\[
1.0000 & -1.3759 & -0.6515 & 1.2847 & -0.3303 \\
-1.3759 & 0.8245 & 3.0770 & -3.7085 & 0.8905 \\
-0.6515 & 3.0770 & -5.6346 & 3.7672 & -0.9966 \\
1.2847 & -3.7085 & 3.7672 & -2.1322 & 0.4965 \\
-0.3303 & 0.8905 & -0.9966 & 0.4965 & -0.1333 \\
\end{array} \right] \times (1 - 1.1761z_1^{-1} + 0.4029z_1^2)(1 - 1.1288z_1^{-1} + 0.7262z_1^2) \times (1 - 1.1761z_1^{-1} + 0.4029z_1^2)(1 - 1.1288z_1^{-1} + 0.7262z_1^2)
\]

The optimal error index \( E^* = 1.0389 \). The maximum errors in passband and stopband are 0.034 and 0.063, respectively. Fig. 2 shows the magnitude response of the filter.

### 8.3 Example 3

A three-dimensional spherically symmetric lowpass filter with the passband and stopband radii as 0.2\( \pi \) and 0.4\( \pi \), respectively, is designed from a fourth-order one-dimensional Butterworth analogue filter

\[
T(s) = \frac{1}{(s^2 + 1.8478s + 1)(s^2 + 0.7654s + 1)}
\]

It is chosen that \( R = S = U = 2 \). A rectangular grid of 286 samples is chosen for optimisation. The resultant

\[<	ext{Figure 3}>\]

The optimal error index \( E^* = 1.2012 \). The maximum errors in the passband and stopband are 0.045 and 0.056, respectively. Fig. 2 shows the magnitude response of the filter.
8.4 Remarks

The proposed design method was implemented on an IBM AT microcomputer (clock rate = 6 MHz) using BASIC language compiled with the Microsoft BASIC Compiler. The computational times of Examples 1, 2 and 3 were 9 hours, 3 hours and 2.5 hours, respectively. As most of the coefficients of the filter are predetermined by the one-dimensional analogue filter(s), the obtained solution is only suboptimal and the precision of the design cannot be very high. However, these moderately precise filters can be used for applications with less stringent specifications or for prefilters.

9 Conclusions

A simple method suitable for M-dimensional ellipsoidal and spherical recursive digital-filter design from a one-dimensional analogue filter has been presented. A special advantage of this method is that the employed one-dimensional analogue filter can be obtained directly from one of the three major types of analogue filters, i.e. Butterworth, Chebyshev and elliptic filters. Owing to the direct utilisation of one-dimensional analogue filters, the number of coefficients required for determination in optimisation can be greatly reduced. Owing to separable denominator, the stability of the filter can also be guaranteed. Moreover, the resultant filter, as expressed in eqn. 7, is quadrantal symmetry for elliptical magnitude response and octagonal symmetry in circular magnitude response. Hence, the number of frequency sample points required for optimisation can be reduced. On the whole, this formulation can be regarded as a simple and suboptimal method for the design of M-dimensional IIR digital filters with moderate precision.

10 References

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