Multidimensional spherically symmetric recursive digital filter design satisfying prescribed magnitude and constant group delay responses

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Abstract: A computationally efficient technique for the design of multidimensional spherically symmetric recursive digital filters satisfying prescribed magnitude and constant group delay specifications is presented. The denominator and the numerator of the transfer function are designed separately. The former is used to approximate the group delay response and the latter is used to approximate the magnitude response. Moreover, this method also makes use of the symmetric conditions of the transfer function. Therefore the numbers of parameters and sample points required for the optimisation are greatly reduced. As a result, the amount of computation can be minimised and the convergence can also be improved. Such advantages are extremely significant for high-order multidimensional filter design. Two examples of the 2-dimensional and one example of the 3-dimensional digital filter design are given to illustrate the proposed method. Comparisons with the results of other methods are also given.

1 Introduction

Two-dimensional and 3-dimensional digital filters are finding applications in many fields such as image processing, seismic signal processing, magnetic data processing and biomedical tomography. In many of these applications, a signal does not have any preferred spatial direction and so the required digital functions can be circularly or spherically symmetric. In the last decade, many methods [1, 2] have been proposed for 2-dimensional recursive digital filter design and some [3–5] for 3-dimensional recursive digital filter design. However, few of them satisfy both magnitude and group delay characteristics in which the latter is extremely important in 2- and 3-dimensional image processing [6].

Maria and Fahmy [7, 8] adopted the idea of using a 2-dimensional digital filter to approximate a given magnitude specification and then cascaded it with a 2-dimensional allpass digital filter to equalise the resulting group delay. However, the overall filter is, in general, not optimal and the filter order becomes higher. Chottera and Jullien [9] formulated this kind of recursive digital filter design as a linear programming problem. However, to use a linear stability constraint, which is only a sufficient condition for stability, the method only yields a subclass of possible solutions. Besides, there are also some methods [10, 11] which tackle this problem by choosing a performance index expressed as a linear combination of three error functions, i.e. one for the magnitude response, one for the group delay response and the latter is used to approximate a given group delay specification. The second part is to design an all-pole digital filter to approximate a given magnitude specification. This method has the advantages of reduction of the amount of calculations, improvement of convergence, stable structure, and reduction in round-off error in cascade realisation. However, the general transfer function of the method does not fully use the properties of circular symmetry. Therefore, in the design of a high-order circularly symmetric filter, the computation time of this method will be very long.

Application of the properties of symmetry [5, 13, 14] in filter design has been tried in References 15 and 16. In this paper, we are going to present a new and computationally efficient method for the design of a multidimensional spherically symmetric digital filter that meets a given set of magnitude and constant group delay specifications. This method adopts a similar procedure to that of Reference 12. However, the transfer function is based on the format of a separable denominator and an inseparable linear phase numerator. Moreover, the properties of octagonal symmetry are fully used in the optimisation procedure. Unlike the method used in Reference 11, the transfer function used consists of a separable denominator and a general inseparable numerator. Other special cases of a 2nd-order 2-dimensional transfer function can also be found in References 15 and 16. Because of the properties of symmetry adopted in our method, the amount of calculation, the number of coefficients to be determined, the number of frequency samples required, and the number of iterations in the optimisation can be greatly reduced. Moreover, the method is further extended and generalised for a multidimensional spherically symmetric digital filter design. This method is extremely suitable for the design of high-order spherically
symmetric digital filters. Two examples of 2-dimensional filter design and one example of 3-dimensional filter design will be used to illustrate this method. Comparisons with the results of two other 2-dimensional filter design methods are also provided.

2 Spherical symmetry properties

Generally speaking, a \( \mathbb{N} \)-dimensional filter \( H(w_1, w_2, \ldots, w_N) \) that possesses spherical symmetry in magnitude means that

\[
|H(w_1, w_2, \ldots, w_N)| = |H(w_1', w_2', \ldots, w_N')|
\]

for all values of \( w_1, w_2, \ldots, w_N \) and \( w_1', w_2', \ldots, w_N' \) in the range \(-\pi < w_i \leq \pi, -\pi < w_i' \leq \pi\), for \( i = 1, 2, \ldots, N \) such that

\[
(w_1^2 + w_2^2 + \cdots + w_N^2)^{1/2} = (w_1'^2 + w_2'^2 + \cdots + w_N'^2)^{1/2}
\]

(2)

Obviously, spherical symmetry must imply the following kinds of symmetry:

\[
|H(w_1, w_2, \ldots, w_i, \ldots, w_N)| = |H(w_1, w_2, \ldots, -w_i, \ldots, w_N)|
\]

(3)

and

\[
|H(w_1, w_2, \ldots, w_i, \ldots, w_j, \ldots, w_N)| = |H(w_1, w_2, \ldots, w_j, \ldots, w_i, \ldots, w_N)|
\]

(4)

for all \( i = 1, 2, \ldots, N \) and \( j = 1, 2, \ldots, N \).

For a \( \mathbb{N} \)-dimensional filter, such kinds of symmetry should be referred as \((2^NN!)\)-hedral symmetry. For example, it should be referred as 8-hedral (octogonal) symmetry for a 2-dimensional filter and 48-hedral symmetry for a 3-dimensional filter [4, 5]. In this paper, a transfer function possessing these kinds of symmetry is chosen to approximate the characteristics of spherical symmetry.

3 Design procedure

In this \( \mathbb{N} \)-dimensional filter design method, the first step is to design a \( \mathbb{N} \)-dimensional all-pole filter to approximate a given group delay specification. Then a linear phase numerator is designed to approximate a given magnitude specification. This method purposely separates the group delay optimisation and the magnitude optimisation to prevent the problems of complex computation and convergency. However, it may lose a certain degree of freedom compared with that of the simultaneous optimisation of group delay and magnitude. Though the denominator is only used to approximate the group delay response, it provides a recursive part for the transfer function so that the magnitude approximation carried out by the numerator can be done more efficiently.

3.1 Group delay approximation

The denominator of the transfer function is chosen to be separable such that the stability problem can be simplified and also the computation can be greatly reduced.

\[
H_d(z_1^{-1}, z_2^{-1}, \ldots, z_N^{-1}) = \frac{1}{D(z_1^{-1}, z_2^{-1}, \ldots, z_N^{-1})}
\]

(5)

where

\[
D(z_1^{-1}, z_2^{-1}, \ldots, z_N^{-1}) = Q(z_1^{-1})Q(z_2^{-1}), \ldots, Q(z_N^{-1})
\]

(6)

\(Q(z_i^{-1})\) is a polynomial in \( z_i^{-1}, z_i^{-1} = \exp(-j\omega_i)\), which can be expressed in the \((2K)\)-th order as

\[
Q(z_i^{-1}) = \prod_{k=1}^{K} (d_0^{(k)} + d_1^{(k)}z_i^{-1} + d_2^{(k)}z_i^{-2})
\]

(7a)

or in the \((2K + 1)\)-th order as

\[
= (d_0^{(k)} + d_1^{(k)}z_i^{-1}) \prod_{k=1}^{K} (d_0^{(k)} + d_1^{(k)}z_i^{-1} + d_2^{(k)}z_i^{-2})
\]

(7b)

To ensure stability, \(d_0^{(k)}\) must be expressed in terms of another set of real nonzero parameters \(\{q_0^{(k)}, p = 0, 1, 2; k = 1, 2, \ldots, K\}\).

It can be verified that the magnitude of eqn. 6 is \((2^NN!)\)-hedrally symmetric. Eqn. 6 can be written as

\[
|H_d(z_1^{-1}, z_2^{-1}, \ldots, z_N^{-1})| = \frac{1}{D(z_1^{-1}, z_2^{-1}, \ldots, z_N^{-1})} \times \exp\left(\theta_d(w_1, w_2, \ldots, w_N)\right)
\]

(9)

where \(\theta_d(w_1, w_2, \ldots, w_N) = \theta'(w_1) + \theta'(w_2) + \cdots + \theta'(w_N)\) is the argument of \(1/Q(z_1^{-1})\), \(i = 1, 2, \ldots, N\). The group delay functions are defined as

\[
\tau_d(w_1, w_2, \ldots, w_N) = -\frac{\partial \theta_d(w_1, w_2, \ldots, w_N)}{\partial w_i}
\]

(11)

Substituting eqn. 11 into eqn. 10, we have

\[
\tau_d(w_1, w_2, \ldots, w_N) = -\frac{\partial \theta_d(w_1, w_2, \ldots, w_N)}{\partial w_i} = \tau(w_i)
\]

(12)

The group delay in the direction of \(w_i\) is a function of a single variable \(w_i\). Moreover, it can easily be verified from eqns. 6, 7 and 12 that

\[
\tau(w_i) = \tau(-w_i)
\]

(13)

and from eqns. 7 and 12

\[
\tau(w_1, w_2, \ldots, w_N) = \tau(w_1, w_2, \ldots, w_N)
\]

(14)

where \(i, j = 1, 2, \ldots, N\).

Based on eqns. 12, 13 and 14, the approximation of group delay is reduced to be a 1-dimensional problem. Hence, the number of sample data for the optimisation of group delay specifications can be greatly reduced. Samples can be chosen only on the \(w_i\) axis in the range \(0 \leq w_i \leq \pi\). Supposing that \(M'\) samples are chosen at \(w_i\)s, \(g = 1, 2, \ldots, M'\) and the corresponding group delays are \(\tau(w_{ig})\). The statement of the problem is to find the parameter vector \(b\)

\[
b = [q_0^{(1)}, q_1^{(1)}, \ldots, q_m^{(1)}, q_0^{(2)}, q_1^{(2)}, \ldots, q_m^{(2)}, \ldots, K]^{T}
\]

(15)

so as to minimise the group delay performance index \(J(b)\) which is defined as

\[
J(b) = \sum_{g=1}^{M'} u_g [\tau(w_{ig}) - \sigma]^2
\]

(16)

where \(\sigma\) is the desired constant group delay, which is chosen to be a positive value [11], and \(u_g\) is a non-

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negative weighting function and is equal to zero outside the passband. The Fletcher–Powell algorithm [17] is used in the optimisation of eqn. 16.

### 3.2 Magnitude approximation

The next step is to design a constant group delay or linear phase numerator polynomial so as to approximate the desired magnitude characteristics. Let the recursive filter be

\[
H(z_1^{-1}, z_2^{-1}, \ldots, z_N^{-1}) = \frac{M(z_1^{-1}, z_2^{-1}, \ldots, z_N^{-1})}{D(z_1^{-1}, z_2^{-1}, \ldots, z_N^{-1})}
\]

(17)

where

\[
M(z_1^{-1}, z_2^{-1}, \ldots, z_N^{-1}) = \prod_{f=1}^{L} M_f(z_1^{-1}, z_2^{-1}, \ldots, z_N^{-1})
\]

(18)

\[M_f(z_1^{-1}, z_2^{-1}, \ldots, z_N^{-1}) = z_1^{-m_f}z_2^{-m_f} \cdots z_N^{-m_f}
\]

\[\times \left( \sum_{a_1=0}^{m_f} \cdots \sum_{a_N=0}^{m_f} C_{a_1 \cdots a_N} \right) \]

\[\times \cos (a_i w_i) \cdots \cos (a_N w_N)
\]

(19)

where

\[C_{a_1 a_2 \cdots a_2 \cdots a_N} = C_{a_1 a_2 \cdots a_N}
\]

\[\cos (a_i w_i) = \frac{z_i^{-a_i} + z_i^{a_i}}{2}, \quad i = 1, 2, \ldots, N
\]

(20)

and the order of the numerator is \(2Lm_f\), where \(m_f\) is a positive integer. Because of the \((2N!^N)\)-hedrally symmetric, the sample data may be taken only in the region \(R\) on the \((w_1, w_2, \ldots, w_N)\) domain such that

\[
R = \begin{cases} 
0 \leq w_1 \leq \pi \\
2 \leq w_2 \\
\vdots \\
w_i \leq w_{i-1} \\
\vdots \\
w_N \leq w_{N-1}, \quad i = 2, 3, \ldots, N
\end{cases}
\]

(22)

Therefore the number of samples required for optimisation can be greatly reduced.

Supposing that \(M^*\) discrete samples \((w_{1m}, w_{2m}, \ldots, w_{Nm})\), for \(m = 1, 2, \ldots, M^*\), in \(R\) are chosen, the corresponding magnitude and desired magnitude are

\[
|H(e^{-jw_{1m}}, e^{-jw_{2m}}, \ldots, e^{-jw_{Nm}})|
\]

and

\[
Y(e^{-jw_{1m}}, e^{-jw_{2m}}, \ldots, e^{-jw_{Nm}})
\]

respectively. For simplicity, they are written as \(|H(m)|\) and \(Y(m)\), respectively. The statement of the problem is to find the parameter vector \(\mathbf{a}\)

\[
\mathbf{a} = [C_{a_1 a_2 \cdots a_N}] a_i = 0, 1, 2, \ldots, m_f;
\]

\[i = 1, 2, \ldots, N; \quad f = 0, 1, 2, \ldots, L\]

(23)

to minimise the magnitude performance index

\[
E(\mathbf{a}) = \sum_{m=1}^{M^*} u_m |H(m)| - Y(m)|^2
\]

(24)

where \(u_m\) is a nonnegative weighting function. The Fletcher–Powell algorithm [17] is used for the optimisation of eqn. 24.

### 4 Examples

Two examples of a 2-dimensional lowpass and bandpass digital filter design and one example of a 3-dimensional lowpass digital filter design are illustrated below. In all examples, for the optimisation of the denominator polynomial, discrete frequency samples are taken at intervals of 0.1\(\pi\) on the \(w_i\) axis within the passband radius. For the optimisation of the numerator polynomial, discrete frequency samples are taken at intervals of 0.1\(\pi\) in the region \(R\) as defined in eqn. 22. For the two examples of the 2-dimensional filter, the weighting functions (Tables 1, 2 and 4) of the frequency samples in both optimisation procedures are chosen to be equivalent to taking at frequency samples at intervals of 0.1\(\pi\) on the \((w_1, w_2)\) plane such that

\[-\pi < w_1 \leq \pi, \quad -\pi < w_2 \leq \pi
\]

(25)

for the purpose of obtaining similar conditions for useful comparison with other methods [10, 12].

The initial value of each unknown coefficient is chosen to be 1. Also for the purpose of comparison, the performance of the filter is shown by the relative root mean square (RMS) errors [12] of the designed filter which are defined as

\[
e_{\text{rms}}(f) = \left( \frac{1}{M^*} \sum_{m=1}^{M^*} u_m |H(m) - Y(m)|^2 \right)^{1/2}
\]

(26)

where

\[S_i = \sigma + \sum_{f=1}^{L} S_i
\]

and

\[
e_{\text{rms}}(f) = \left( \frac{1}{M^*} \sum_{m=1}^{M^*} |H(m) - Y(m)|^2 \right)^{1/2}
\]

(27)

#### 4.1 Examples of 2-dimensional filter design

In the \((w_1, w_2)\) plane, actually, the region \(R\) is the \([0^\circ, 45^\circ]\) sector. Therefore, samples are taken in this sector for the optimisation of the magnitude specifications. The two examples shown below have been tried in References 10 and 12. Comparisons of the performance of this method with the design methods mentioned in References 10 and 12 are also provided.

#### 4.1.1 Example 1: Lowpass filter: This example is to design a 2-dimensional circularly symmetric lowpass filter with the following desired magnitude specifications:

<table>
<thead>
<tr>
<th>(w_1)</th>
<th>(w_2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0 \leq w_1 \leq \pi)</td>
<td>(0 \leq w_2 \leq \pi)</td>
</tr>
</tbody>
</table>

respectively. For simplicity, they are written as \(|H(m)|\) and \(Y(m)\), respectively. The statement of the problem is to find the parameter vector \(\mathbf{a}\)

\[
\mathbf{a} = [C_{a_1 a_2 \cdots a_N}] a_i = 0, 1, 2, \ldots, m_f;
\]

\[i = 1, 2, \ldots, N; \quad f = 0, 1, 2, \ldots, L\]

(23)

to minimise the magnitude performance index

\[
E(\mathbf{a}) = \sum_{m=1}^{M^*} u_m |H(m)| - Y(m)|^2
\]

(24)

where \(u_m\) is a nonnegative weighting function. The Fletcher–Powell algorithm [17] is used for the optimisation of eqn. 24.
and a single section \((L = 1)\) 4th-order numerator \((m_f = 2)\). The weighting function \(u_m\) is chosen according to Table 1 and the weighting function \(u_g\) is chosen according to Table 2.

### Table 1: Weighting functions \(u_m\) of the samples used in the examples of 2-dimensional filter design

<table>
<thead>
<tr>
<th>(z)</th>
<th>(u_m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.9(\pi)</td>
</tr>
<tr>
<td>4</td>
<td>0.8(\pi)</td>
</tr>
<tr>
<td>4</td>
<td>0.7(\pi)</td>
</tr>
<tr>
<td>4</td>
<td>0.6(\pi)</td>
</tr>
<tr>
<td>4</td>
<td>0.5(\pi)</td>
</tr>
<tr>
<td>4</td>
<td>0.4(\pi)</td>
</tr>
<tr>
<td>4</td>
<td>0.3(\pi)</td>
</tr>
<tr>
<td>4</td>
<td>0.2(\pi)</td>
</tr>
<tr>
<td>4</td>
<td>0.1(\pi)</td>
</tr>
<tr>
<td>1</td>
<td>0.0 (\pi)</td>
</tr>
</tbody>
</table>

Table 2: Weighting function \(u_g\) for example 1

\[
g \quad \omega \quad u_g \\
1 \quad 0 \quad 7 \\
2 \quad 0.1\pi \quad 10 \\
3 \quad 0.1\pi \quad 10 \\
4 \quad 0.1\pi \quad 2 \\
\]

After 32 iterations, the resultant denominator is

\[
D(z^{-1}, z^{-2}) = Q(z_1^{-1})Q(z_2^{-1})
\]

where

\[
Q(z_i^{-1}) = (1.670937 - 1.503865z_i^{-1} + 0.825198z_i^{-2}) \times (1.577846 - 1.841585z_i^{-1} + 0.580569z_i^{-2}), \quad i = 1, 2
\]

After 28 iterations, the resultant numerator is

\[
N(z_1^{-1}, z_2^{-1}) = 10^{-2}[1, z_1^{-1}, z_1^{-2}, z_1^{-3}, z_1^{-4}] \times A[1, z_2^{-1}, z_2^{-2}, z_2^{-3}, z_2^{-4}]^T
\]

where

\[
A = \begin{bmatrix}
2.255632 & -2.910392 \\
-2.910392 & 5.625357 \\
3.282184 & -4.034687 \\
-2.910392 & 5.625357 \\
2.255632 & -2.910392 \\
-4.034687 & 5.625357 \\
4.716775 & -4.034687 \\
3.282184 & -2.910392 \\
2.255632 & -2.910392 \\
\end{bmatrix}
\]

In this example, it was found that the resultant denominator was identical to that given in Reference 12. In this case, the performance indices eqns. 16 and 24 have the values \(J(b) = 1.75 \times 10^{-9}\) and \(E(a) = 0.158\), respectively.

### Table 3: Results of example 1

<table>
<thead>
<tr>
<th>(\varepsilon_{ti})</th>
<th>(\varepsilon_{t1})</th>
<th>(\varepsilon_{m})</th>
<th>Number of independent parameters</th>
<th>Number of samples required (denominator)</th>
<th>Number of samples required (numerator)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.94 \times 10^{-4}</td>
<td>1.94 \times 10^{-4}</td>
<td>12.81</td>
<td>10</td>
<td>66</td>
<td>4</td>
</tr>
<tr>
<td>1.94 \times 10^{-4}</td>
<td>1.94 \times 10^{-4}</td>
<td>16.95</td>
<td>29</td>
<td>220</td>
<td>17</td>
</tr>
<tr>
<td>9.32</td>
<td>8.18</td>
<td>24.36</td>
<td>33</td>
<td>220</td>
<td>17</td>
</tr>
</tbody>
</table>

Table 3 shows the comparison of the performance of the present method with the two other methods reported in References 10 and 12. It can be seen from Table 3 that the present method is better in both error performance as well as efficiency of optimisation. The group delay and magnitude responses of the filter are shown in Fig. 1.

**Fig. 1** Passband group delay and magnitude response of the filter given in example 1

\(a\) \(b\) \(c\)

### 4.1.2 Example 2: Bandpass filter

This example is to design a 2-dimensional circularly symmetric bandpass filter with the desired magnitude specifications

\[Y(\exp (-jw_1), \exp (-jw_2)) = (E/(E^2 + 1/S))^{1/2}\]

where

\[E = (\pi/c)^{1/2} \exp (-r^2/c) \quad r = (w_1^2 + w_2^2)^{1/2}\]

\[S = 100 \quad c = 0.5\]
The group delay is required to be constant in the passband where \( r > 0.6\pi \), (see Appendix).

Let the desired group delay \( \sigma = 1 \). The filter is chosen to consist of a 4th-order (\( K = 2 \)) denominator (see eqn. 7a) and a single section (\( L = 1 \)) 4th-order numerator (\( m_f = 2 \)). The weighting function \( u_m \) is chosen according to Table 1 and the weighting function \( u_g \) is chosen according to Table 4.

Table 4: Weighting function \( u_g \) for example 2

<table>
<thead>
<tr>
<th>( g )</th>
<th>( \omega_e )</th>
<th>( u_g )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>0.1\pi</td>
<td>22</td>
</tr>
<tr>
<td>3</td>
<td>0.2\pi</td>
<td>22</td>
</tr>
<tr>
<td>4</td>
<td>0.3\pi</td>
<td>22</td>
</tr>
<tr>
<td>5</td>
<td>0.4\pi</td>
<td>18</td>
</tr>
<tr>
<td>6</td>
<td>0.5\pi</td>
<td>14</td>
</tr>
</tbody>
</table>

After 24 iterations, the resultant denominator is

\[
D(z_1^{-1}, z_2^{-2}) = Q(z_1^{-1})Q(z_2^{-1})
\]

where

\[
Q(z_i^{-1}) = (2.408485 - 0.599085z_i^{-1} + 0.992430z_i^{-2}) \\
\times (1.672211 - 1.752135z_i^{-1} + 0.575655z_i^{-2}), \quad i = 1, 2
\]

After 18 iterations, the numerator obtained is

\[
M(z_1^{-1}, z_2^{-1}) = [1, z_1^{-1}, z_1^{-2}, z_1^{-3}, z_1^{-4}]A[1, z_2^{-1}, z_2^{-2}, z_2^{-3}, z_2^{-4}]^T
\]

where

\[
A =
\begin{bmatrix}
0.188264 & 0.682142 \\
0.682142 & -0.763777 \\
0.025492 & -0.607780 \\
0.682142 & -0.763777 \\
0.188264 & 0.682142 \\
-0.607780 & -0.763777 \\
-0.607780 & -0.763777 \\
-0.607780 & -0.763777 \\
0.205492 & 0.188264
\end{bmatrix}
\]

In this case, the performance indices eqns. 16 and 24 have the values \( J(b) = 5.32 \) and \( E(a) = 7.36 \), respectively. Table 5 shows the comparison of the performance of this method with the other two methods. It can be seen from Table 5 that the error performance indices of this method and the method in Reference 12 are approximately equal but the efficiency of the optimisations of this method is much better. The group delay and magnitude responses of the filter are shown in Fig. 2.

4.2 Example of 3-dimensional filter design

4.2.1 Example 3: Lowpass filter: This example is to design a 3-dimensional spherically symmetric lowpass filter with the desired magnitude specifications as example 1 with \( r = (w_1^2 + w_2^2 + w_3^2)^{1/2} \). Also the group delay is required to be constant in the passband, \( r \leq 0.3\pi \).

Table 5: Results of example 2

<table>
<thead>
<tr>
<th>( \varepsilon_{e_1} )</th>
<th>( \varepsilon_{e_2} )</th>
<th>( \varepsilon_{e_m} )</th>
<th>Number of independent parameters</th>
<th>Number of samples required (numerator)</th>
<th>Number of samples required (denominator)</th>
</tr>
</thead>
<tbody>
<tr>
<td>7.36</td>
<td>7.36</td>
<td>18.07</td>
<td>10</td>
<td>66</td>
<td>4</td>
</tr>
<tr>
<td>7.20</td>
<td>7.20</td>
<td>18.45</td>
<td>29</td>
<td>220</td>
<td>60</td>
</tr>
<tr>
<td>15.11</td>
<td>14.62</td>
<td>58.86</td>
<td>33</td>
<td>220</td>
<td>60</td>
</tr>
</tbody>
</table>

Fig. 2 Passband group delay and magnitude response of the filter given in example 2

Let the desired group delay \( \sigma = 2 \). The filter is chosen to have a 4th-order (\( K = 2 \)) denominator (see eqn. 7a) and a single section (\( L = 1 \)) 4th-order numerator (\( m_f = 2 \)). In this example, all the weighting functions, \( u_g \) and \( u_m \) are chosen to be 1. After 24 iterations, the resultant denominator is

\[
D(z_1^{-1}, z_2^{-2}) = Q(z_1^{-1})Q(z_2^{-1})Q(z_3^{-1})
\]

where

\[
Q(z_i^{-1}) = (1.577846 - 1.841585z_i^{-1} + 0.580569z_i^{-2}) \\
\times (1.670937 - 1.503865z_i^{-1} + 0.825198z_i^{-2}), \quad i = 1, 2, 3
\]
After 40 iterations, the coefficients of the numerator are
\[ C_{0,0,0} = -1.733318 \times 10^{-1} \]
\[ C_{0,0,1} = 1.109532 \times 10^{-1} \]
\[ C_{0,0,2} = -2.038416 \times 10^{-2} \]
\[ C_{0,1,1} = -1.280308 \times 10^{-1} \]
\[ C_{0,1,2} = 4.743257 \times 10^{-3} \]
\[ C_{0,2,2} = 1.866163 \times 10^{-2} \]
\[ C_{1,1,1} = 3.097474 \times 10^{-2} \]
\[ C_{1,1,2} = 7.832095 \times 10^{-3} \]
\[ C_{1,2,2} = -5.384163 \times 10^{-2} \]
\[ C_{2,2,2} = 1.144929 \times 10^{-2} \]

In this case, the performance indices eqns. 16 and 24 have the values \( t_0 = 2.5 \times 10^{-22} \) and \( E(a) = 0.0877 \), respectively. Table 6 shows the relative RMS errors. The group delay and magnitude responses are shown in Fig. 3 which are very satisfactory.

### 4.3 Remarks

The proposed design method was implemented on an IBM PC/XT microcomputer (clock rate = 4.77 MHz) using Basic compiled with the Microsoft Basic Compiler. The computation times of examples 1, 2 and 3 were 74 minutes, 44 minutes and 10 hours, respectively. However, these computation times can be reduced considerably if an IBM PC/AT microcomputer and a more efficient language are used.

The results of this method shown in Tables 3 and 5 have some slight differences compared with those of the values obtained by Reference 12. It may be due to the difference in the wordlengths of the computers used. However, such minor differences are reasonable. The results obtained using the method in Reference 10 shown in Tables 3 and 5 are obtained directly from Tables 1 and 2 of Reference 12.

For a special case of \( m_f = 1 \), the transfer function eqns. 6, 7 and 17-19 can then be written as a cascade of 1st- and 2nd-order modules. This format is then very suitable for modular implementation.

### 5 Conclusions

In this paper, a computationally efficient method has been presented for the design of a multidimensional spherically symmetric recursive digital filter to approximate a given set of magnitude and constant group delay specifications. The transfer function is inherently stable and \( (2^n)^n \)-hedrally symmetric. Therefore the number of independent coefficients required to be determined and the number of samples to be used in the optimisations can be greatly reduced. To test the performance of this method, two 2-dimensional examples used in References 10 and 12 are used in this paper. This method generally shows better performance in error indices and, especially, the efficiency of optimization when compared with results of previous papers. Moreover, a 3-dimensional example has also been tried and gives good results. Based on the results obtained, one can conclude that this method is a reliable and computationally efficient method for the design of multidimensional spherically symmetric recursive digital filter. Furthermore, the efficiency increases as the order of the digital filter to be designed increases.

### 6 Acknowledgment

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### 7 References

5. **PITAS, I., and VENETSANOPULOUS, A.N.** 'The use of symmetries in the design of multidimensional digital filters', *ibid.*, 1986, CAS-33, pp. 863-873

### 8 Appendix

In example 2, the constant group delay specification of the 2-dimensional bandpass filter covers the portion of
the stopband which includes the origin. The reason is explained below.

According to the properties of symmetry (eqns. 12, 13 and 14) of the transfer function, for a particular frequency \( w' \),

\[
\tau(w', 0) = a
\]

is equivalent to

\[
\tau(w', w_2) = \tau(-w', w_2) = \tau(w_1, w') = \tau(w_1, -w') = a
\]

for \(-\pi < w_1 < \pi\) and \(-\pi < w_2 < \pi\). Therefore to specify that the passband of the bandpass filter has a constant group delay is equivalent to specifying that the shaded region in Fig. 4 be of constant group delay, that is to say, \( \tau(w) \) is constant for \( 0 \leq w_1 \leq w_c \).

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**Fig. 3** Passband group delay \( \tau(w_1, w_2) + 2 \) in the \((w_1, w_2)\) plane and magnitude responses of the filter given in example 3

a. Passband group delay, \(-\pi < w_2 < \pi\)

b. Magnitude response with \( w_2 = 0 \)

c. Magnitude response with \( w_2 = w_1 \)

**Fig. 4** Region of constant group delay in specifying \( \tau(w_1) \), \( 0 \leq w_1 \leq w_c \), to be a constant

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