IIR Digital Filter Design with New Stability Constraint Based on Argument Principle

Aimin Jiang and Hon Keung Kwan, Senior Member, IEEE

Abstract—This paper presents a weighted least-squares (WLS) method for IIR digital filter design using a new stability constraint. Utilizing the reweighting technique, an iterative second-order cone programming (SOCP) method is employed to solve the design problem, such that either linear or second-order cone (SOC) constraints can be further incorporated. In order to guarantee the stability of designed IIR digital filters, a new stability constraint with a prescribed pole radius is derived from the argument principle of complex analysis. As compared with other frequency-domain stability constraints, the argument principle is both sufficient and necessary. Since the derived stability constraint cannot be directly incorporated in the iterative SOCP method, the similar reweighting technique is deployed to approximate the stability constraint in a quadratic form, which is then combined with the WLS iterative design process. Filter design examples are presented to demonstrate the effectiveness of the proposed iterative SOCP method.

Index Terms—Argument principle, infinite impulse response (IIR) digital filters, reweighting techniques, second-order cone programming (SOCP), weighted least-squares (WLS) approximation.

I. INTRODUCTION

Compared with FIR digital filter design, the major difficulties for designing an IIR digital filter are its nonlinearity and stability problems. Many algorithms have been developed to implement stable IIR digital filters. Some approaches [1]–[5] implement filters in an indirect way, that is, an FIR digital filter satisfying the filter specifications are designed first, and then model reduction techniques are applied to approximate the FIR digital filter by a reduced-order IIR digital filter. In such indirect designs, approximation procedures can substantially guarantee the stability of designed IIR digital filters, which facilitates the design procedures. However, it is difficult to design filters with accurate cutoff frequencies using this design strategy. Recently, many other algorithms [6]–[15] have been proposed to design IIR digital filters in a direct way, which means the cost function of the design problem is directly based on the ideal frequency responses. In order to tackle such a nonlinear design problem, some iterative methods [7]–[11] employ the Steiglitz-McBride scheme [16] to approach the optimal solution in some sense. Although so far the convergences of these methods cannot be strictly guaranteed, many examples in literatures have shown their effectiveness. In this paper, we adopt the similar iterative procedure using the Steiglitz-McBride scheme to design IIR digital filters.

Stability is an important issue for IIR digital filter design. Recently, some positive-realness [6], [7], [9]–[12] and the Rouché’s theorem [13] based frequency-domain stability constraints have been developed and utilized in the design of stable IIR digital filters. Positive-realness [12] based stability constraint is expressed as linear inequalities with respect to denominator coefficients, which can be easily incorporated in linear and quadratic programming problems. The Rouché’s theorem based stability constraint [13], which is less conservative than the positive-realness based stability constraint, is expressed as quadratic inequalities in terms of denominator coefficients. Note that both of them should be satisfied at all frequencies $\omega \in [0, \pi]$. A simple way [7], [9]–[12] to incorporate these constraints is to realize them on a set of dense grid frequency points over $[0, \pi]$. The number of stability constraints, therefore, is large. Another way is to employ a multiple exchange algorithm [13], which is used to identify a set of active constraints. Then stability constraints are replaced by a finite number of active constraints. In [6], by defining a convex stability domain, the design problem can be formulated as an iterative semidefinite programming (SDP) problem. It has been proved in [6] that this stability domain contains the domain given by the Rouché’s theorem. Although the effectiveness of stability constraints described above has been proved by theoretical analyses and demonstrated by examples therein, they are merely sufficient conditions for stability. This means that some stable filters can be excluded from their admissible solutions. Recently, a stability constraint based on the argument principle of complex analysis has been introduced in [14], which is both sufficient and necessary. By truncating the higher-order expansion components, the resulting stability constraint becomes a linear equality constraint. However, through a large number of simulations, this constraint could be found to be invalid in some situations. As an attempt to solve this problem, a new stability constraint was proposed which is also based on the argument principle. Unlike [14], the stability constraint is approximated in a quadratic form, which is then combined with the iterative design procedure. It will be shown that this stability constraint is imposed on an integral over the whole frequency band $[0, \pi]$. Hence, neither a large number of

Manuscript received October 25, 2007, revised April 30, 2008. This work was supported in part by the Natural Sciences and Engineering Research Council of Canada.

The authors are with the Department of Electrical and Computer Engineering, University of Windsor, Windsor, Ontario, Canada N9B 3P4 (e-mail: jiang13@uwindsor.ca; kwan1@uwindsor.ca).

Copyright (c) 2008 IEEE. Personal use of this material is permitted. However, permission to use this material for any other purposes must be obtained from the IEEE by sending an email to pubs-permissions@ieee.org.
constraints on dense grid frequency points nor multiple exchange algorithms are needed.

In practice, two optimal criteria, minimax \((L_\infty)\) [7], [8], [12], [14] and (weighted) least-squares \((L_2)\) [6], [9], [13], [14], are most frequently used in IIR filter design. Some other criteria, such as equiripple passband and peak-constrained least-squares stopband (EPPCLSS) [10], least \(p\)-power error minimization [11], and weighted integral of the squared error (WISE) [15], are also applied in IIR digital filter design. In this paper, we will focus on IIR filter design in the weighted least-squares (WLS) sense.

Convex optimization [17] methods, such as second-order cone programming (SOCP) [17], [18] and SDP [17], [19], have been widely applied in designing FIR [20]-[24] and IIR digital filters [1], [6], [8]. Compared with linear programming (LP) and convex quadratic programming (QP) problems, SOCP problems can incorporate more general constraints besides linear equalities and inequalities. As a matter of fact, LP and convex QP problems are special cases of SOCP problems. Therefore, all the LP and convex QP problems can also be solved by the SOCP algorithms. Although SOCP problems are less general than SDP problems, the computation complexity per iteration required in the SOCP algorithms is much less than that in the SDP algorithms, especially when the dimensions of second-order cone (SOC) constraints are large. Therefore, we will cast the IIR filter design problem into the SOCP form. In [25], we have presented a preliminary version of this paper. In order to incorporate the quadratic peak constraints, the design method shall be implemented in this paper as an iterative SOCP procedure instead of the iterative QP procedure adopted in [25]. Accordingly, the quadratic cost function in [25] is cast into an SOC constraint. Since the stability constraint derived from the argument principle cannot be directly incorporated in the design procedure, an approximated stability constraint is further proposed in this paper. Two new examples and further discussions, here we first define a weighted complex error as

\[
E(\omega) = W(\omega) \left[ D(\omega) - H(e^{j\omega}) \right] = W(\omega) \left[ D(\omega) - \frac{P(e^{j\omega})}{Q(e^{j\omega})} \right], \quad \omega \in \Omega, \tag{2}
\]

where \(W(\omega)\) is a given nonnegative weighting function, and \(\Omega\) is the union of frequency bands of interest over \([0, \pi]\).

B. Weighted Least-Squares Design

In the WLS sense, the design problem can be expressed as

\[
\min_{\Omega} \int_{\Omega} |E(\omega)|^2 \, d\omega \tag{3}
\]

By introducing an auxiliary variable \(\delta\), (3) can be formulated as

\[
\begin{align*}
\min_{\Omega} & \quad \delta \\
\text{s.t.} & \quad \int_{\Omega} |E(\omega)|^2 \, d\omega \leq \delta^2 \tag{4}
\end{align*}
\]

Because of the existence of denominator \(Q(e^{j\omega})\) in \(E(\omega)\), the constraint of the design problem (4) cannot be cast into an either linear or SOC constraint. Here we consider an iterative scheme which has been deployed in [7]-[11]. At the \(k\)th iteration, the constraint of (4) is written as

\[
\int_{\Omega} W_2^{(k-1)}(\omega) \left| D(\omega)Q^{(k)}(e^{j\omega}) - P^{(k)}(e^{j\omega}) \right|^2 \, d\omega \leq \delta^2 \tag{5}
\]

where

\[
W_2^{(k-1)}(\omega) = \frac{W^2(\omega)}{\left| Q^{(k-1)}(e^{j\omega}) \right|^2} \tag{6}
\]

Note that, by introducing

\[
E^{(k)}(\omega) = W(\omega) \left[ D(\omega) - \frac{P^{(k)}(e^{j\omega})}{Q^{(k)}(e^{j\omega})} \right] \tag{7}
\]
\[ s^{(k)}(\omega) = \frac{Q^{(k)}(e^{j\omega})}{Q^{(k-1)}(e^{j\omega})} \]  

(8)

at the \( k \)th iteration, the integrand in the constraint (5) can be expressed as \(|E^{(k)}(\omega)\delta^{(k)}(\omega)|^2\). Obviously, if the iterative procedure converges, then \( s^{(k)}(\omega) \) will approach 1 as \( k \) is large enough. Thus, the integrand will approach \(|E^{(k)}(\omega)|^2\).

By defining

\[
\begin{align*}
\mathbf{x}^{(k)} &= \begin{bmatrix} q^{(k)} \\ p^{(k)} \end{bmatrix} = \mathbf{x}^{(k-1)} + \gamma \mathbf{\eta}^{(k)}, \quad 0 < \gamma < 1, \quad \mathbf{\eta}_0^{(k)} = 0 \\
\mathbf{c}(\omega) &= \begin{bmatrix} D(\omega)\mathbf{\varphi}_1(e^{j\omega}) \\ -\mathbf{\varphi}_1(e^{j\omega}) \end{bmatrix}
\end{align*}
\]

(9)

(10)

where \( \mathbf{x}^{(k-1)} \) is the coefficient vector obtained at the previous iteration, \( \gamma \) is the step size, and \( \mathbf{\eta}^{(k)} \) is the updating vector at the current iteration, the constraint in (5) is rewritten as

\[
\mathbf{A}^{(k-1)} \mathbf{x}^{(k)} \leq \delta^2
\]

(11)

where

\[
\mathbf{A}^{(k-1)} = \int_{\Omega} W_q^{(k-1)}(\omega) \cdot \text{Re}\{\mathbf{c}(\omega)\mathbf{c}^H(\omega)\} \, d\omega
\]

(12)

In (12), the superscript \( H \) denotes the complex conjugate and transpose operation, and \( \text{Re}\{\cdot\} \) represents the real part of a complex. Since \( \mathbf{A}^{(k-1)} \) in (12) is a positive definite matrix, the design problem (4) with the constraint (11) can be further cast into an SOCP problem

\[
\begin{align*}
\min_{\mathbf{\eta}^{(k)}} & \quad \delta \\
\text{s.t.} & \quad \mathbf{\eta}_0^{(k)} = 0 \\
& \quad \|\mathbf{F}^{(k-1)}\mathbf{\eta}^{(k)} + \mathbf{g}^{(k-1)}\| \leq \delta
\end{align*}
\]

(13)

where

\[
\begin{align*}
\mathbf{F}^{(k-1)} &= \gamma \left[ \mathbf{A}^{(k-1)} \right]^\frac{1}{2} \\
\mathbf{g}^{(k-1)} &= \left[ \mathbf{A}^{(k-1)} \right]^\frac{1}{2} \mathbf{x}^{(k-1)}
\end{align*}
\]

(14)

(15)

The norm appearing in the constraint of (13) is the standard Euclidean norm, i.e., \( |\mathbf{u}| = (\mathbf{u}^H\mathbf{u})^{1/2} \).

The iterative procedure stops if the following condition is satisfied

\[
\|\mathbf{\eta}^{(k)}\| \leq \varepsilon
\]

(16)

where \( \varepsilon \) is a prescribed small tolerance, or \( k \) exceeds the specified maximum number of iterations. Otherwise, the new coefficient vector is computed according to (9).

Although the convergence of the iterative procedures has not been strictly guaranteed, a sufficient condition for convergence has been provided in [9] and [11]. Numerical examples presented in [7]-[11] also demonstrate the effectiveness of this design strategy.

C. Peak Constraint

In [7] and [11], linearized peak constraints have been developed to control the peak errors. Here, we will reformulate the peak constraints as SOC constraints, which can better approximate the true peak constraints.

Using (2), the peak constraints are given by

\[
|E(\omega)| = \left| D(\omega) - \frac{P(e^{j\omega})}{Q(e^{j\omega})} \right| \leq \mu_i, \quad \omega_i \in \Omega, \quad i = 1, 2, \ldots, K
\]

(17)

where \( \mu_i \) denotes the prescribed peak error limit. Like the difficulty encountered in formulating the design problem (4), the true peak constraint also has the denominator on the left hand side of (17). Adopting the similar reweighting technique employed in (5), we obtain

\[
\left| D(\omega)Q^{(k)}(e^{j\omega}) - P^{(k)}(e^{j\omega}) \right| \\
\leq \mu_i |Q^{(k-1)}(e^{j\omega})|, \quad \omega_i \in \Omega, \quad i = 1, 2, \ldots, K
\]

(18)

where

\[
\mathbf{B}(\omega) = \left[ \text{Re}\{\mathbf{c}(\omega)\mathbf{c}^H(\omega)\} \right]^\frac{1}{2}
\]

(19)

Note that in [7] and [11] the IIR filter design problems are cast into LP and QP forms, in which only linear constraints can be incorporated. Therefore, the approximation of a circle by some regular polygons is applied to linearize the constraint (18). Although this approximation is applicable when the edge number of regular polygons is large enough, the total number of peak constraints is rapidly increased. Moreover, this step may introduce some approximation error.

III. ARGUMENT PRINCIPLE BASED STABILITY CONSTRAINT

In this section, a new stability constraint based on the argument principle from complex analysis is developed. First of all, the argument principle is reviewed. The stability constraint derived from the argument principle is then approximated by a quadratic constraint and combined with the iterative design procedure mentioned in Section II.

A. Argument Principle

If \( f(z) \) is analytic in a region \( R \) enclosed by a contour \( C \) in the \( z \)-plane except at a finite number of poles, let \( N_z \) be the number of zeros and \( N_p \) be the number of poles of the function \( f(z) \) in \( R \), where each zero and pole is counted according to its
multiplicity. Then we have

$$N_z - N_p = \frac{1}{2\pi j} \oint_C \frac{f'(z)dz}{f(z)}$$  \hspace{1cm} (20)$$

This result is called the argument principle [26].

In order to develop a stability constraint for IIR digital filter designs, we consider the following monic polynomial function

$$f(z) = z^M Q(z) = \sum_{n=0}^{M} q_n z^{M-m}, \quad q_0 = 1$$  \hspace{1cm} (21)$$

Here, the contour $C$ is an origin-centered circle with the prescribed radius $r$, i.e., $C = \{ z : |z| = r, r < 1 \}$. Then according to the argument principle stated above, all the zeros of $f(z)$ lie strictly in the region $R$ enclosed by $C$, if and only if the following equation is satisfied

$$M = \frac{1}{2\pi j} \oint_C \frac{f'(z)dz}{f(z)}$$  \hspace{1cm} (22)$$

The integral in (22) is carried out counterclockwise along $C$. Note that

$$\oint_C \frac{f'(z)dz}{f(z)} = \oint_C d\ln f(z)$$  \hspace{1cm} (23)$$

$$= \oint_C d\ln |f(z)| + \oint_C d\arg f(z)$$

where $\arg f(z)$ denotes the argument of $f(z)$. The first term on the right-hand side of (23) is always equal to zero, since $|f(z)|$ is an even function of $\omega$ and $C$ is closed. According to (21), the argument of $f(z)$ can be expanded as $M \omega + \arg Q(z)$ on $C$, i.e., $z = re^{i\omega}$, and then the stability constraint (22) can be simplified by

$$M = \frac{1}{2\pi j} \oint_C d\arg Q(z) = 0$$  \hspace{1cm} (24)$$

Thus, the stability constraint (24) of an IIR digital filter is stated as: An IIR digital filter with the denominator $Q(z)$ is stable, if and only if the total change in the argument of $Q(z)$ is equal to 0, when the integral is carried out along $C$ counterclockwise.

B. Stability Constraint

The polynomial function $Q(re^{i\omega})$ can be expressed as

$$Q(re^{i\omega}) = Q(z)|_{z=re^{i\omega}} = Q_e(re^{i\omega}) + jQ_i(re^{i\omega})$$  \hspace{1cm} (25)$$

where

$$Q_e(re^{i\omega}) = \text{Re}\left\{Q(re^{i\omega})\right\}$$  \hspace{1cm} (26)$$

$$Q_i(re^{i\omega}) = \text{Im}\left\{Q(re^{i\omega})\right\}$$  \hspace{1cm} (27)$$

In (27), $\text{Im}\{\cdot\}$ represents the imaginary part of a complex. The argument of $Q(re^{i\omega})$ is then computed by

$$\arg Q(re^{i\omega}) = \arctan \frac{Q_i(re^{i\omega})}{Q_e(re^{i\omega})}$$  \hspace{1cm} (28)$$

By taking differentials with respect to $\omega$ on both sides of (28), we obtain

$$\frac{d}{d\omega} \arg Q(re^{i\omega}) = \frac{q^T \Psi(re^{i\omega}) q}{|Q(re^{i\omega})|^2}$$  \hspace{1cm} (29)$$

where

$$A = \text{diag} \{0, 1, \ldots, M\}$$  \hspace{1cm} (30)$$

$$\Psi(re^{i\omega}) = \text{Re}\left\{\varphi_1(re^{i\omega}) \varphi_2(re^{i\omega})\right\}$$

$$= \begin{bmatrix} 1 & r^{-1} \cos \omega & \cdots & r^{-M} \cos M\omega \\ r^{-1} \cos \omega & r^{-2} & \cdots & r^{-(M+1)} \cos(M-1)\omega \\ \vdots & \ddots & \ddots & \vdots \\ r^{-M} \cos M\omega & r^{-M-1} \cos(M-1)\omega & \cdots & r^{-2M} \end{bmatrix}$$  \hspace{1cm} (31)$$

In (30), $\text{diag}\{a_0, a_1, \ldots, a_M\}$ denotes a diagonal matrix with $a_i$ $(i = 0, 1, \ldots, M)$ on its $i$th diagonal. By computing the integral of (24) over $[0, \pi]$, the stability constraint (24) is strictly expressed as

$$\tau(r, q) = q^T G(r, q) q = 0$$  \hspace{1cm} (32)$$

where

$$G(r, q) = \int_0^\pi A \Psi(re^{i\omega}) \frac{d\omega}{|Q(re^{i\omega})|^2}$$  \hspace{1cm} (33)$$

On the other hand, if there are $L$ $(\leq M)$ roots of $Q(z)$ outside $C$ and $(M-L)$ roots inside $C$, it can be easily verified that $\tau(r, q) = L\pi$. Then, given a denominator $q$, $\tau(r, q)$ has a stair shape with respect to $r$.

Equation (32) cannot be incorporated directly as a constraint into the design problem of (13), since $\tau(r, q)$ is neither linear form nor SOC form of $q$. In our design, we construct a new stability constraint as

$$\tau(r, q) = q^T G(r, q) q \leq \rho$$  \hspace{1cm} (34)$$

Decreasing $\rho$ makes more poles move inside the circle $C$. When $0 < \rho < \pi$, all poles will lie inside $C$.

An obstacle encountered in constructing the stability criterion (34) is the existence of denominator depending on $q$ in $G(r, q)$. Here, we adopt the similar iterative strategy mentioned in Section II. At the $k$th iteration, $\tau(r, q)$ is modified as
\[ \tau(q^{(k)}) = q^{(k)} \bar{A}(q^{(k)}) q^{(k)} \]  

(35)

As the iterative procedure converges, it follows that \( s^{(k)}(\omega) \) \( \approx 1 \). Then, we can obtain that

\[
G(r, q^{(k-1)}) = \int_0^\pi \frac{A\Psi\left(r e^{j\omega}\right)}{\left[\bar{A}(q^{(k-1)})\right]^2} \left| s^{(k)}(\omega) \right|^2 d\omega \\
= G(r, q^{(k)})
\]

Thus, the stability constraint \( \tau(r, q^{(k)}) \approx q^{(k)} G(r, q^{(k)}) q^{(k)} \leq \rho < \pi \) can guarantee the stability of the IIR digital filter obtained at the \( k \)th iteration as \( k \) is large enough. Note that since \( G(r, q^{(k-1)}) \) is an indefinite matrix, this explicit stability constraint cannot be transformed into an SOC constraint. Therefore, we combine the stability constraint with the constraint (11):

\[ x^{(k)} \bar{A}(q^{(k)}) x^{(k)} \leq \delta^2 \]  

(36)

where

\[ \bar{A}^{(k-1)} = A^{(k-1)} + (1-\alpha) \hat{G}(r, q^{(k-1)}), \quad 0 < \alpha < 1 \]  

(37)

\[ \hat{G}(r, q^{(k-1)}) = \begin{bmatrix} G(r, q^{(k-1)}) & 0_{M(M+1)}(N+1) \\ 0_{(N+1)(M+1)} & 0_{(N+1)(N+1)} \end{bmatrix} \]  

(38)

In (38), \( \theta_{m,n} \) represents a zero matrix of size \( m \)-by- \( n \). Then \( A^{(k-1)} \) in (14) and (15) is replaced by \( \bar{A}^{(k-1)} \). Since \( \tau(r, q) \) increases rapidly when any pole leaves the prescribed region \( R \), \( \tau(r, q) \) essentially serves as a barrier function in (36). This design strategy has been utilized in [15] for the IIR digital filter design. In [15], the authors construct the constraint function by linearly combining the WLS criterion and time-domain component which serves as the implicit stability constraint. In practice, we can decrease \( \alpha \) to achieve lower \( \tau(r, q) \), which corresponds to decreasing \( \rho \) of (34) as \( k \to \infty \). Therefore, besides the prescribed maximum pole radius \( r \), the parameter \( \alpha \) also plays an important role of restricting poles’ locations. On the other hand, increasing \( \alpha \) makes \( \bar{A}^{(k-1)} \) approach an indefinite matrix, which cannot be used to formulate the SOC constraint in (13). Therefore, \( \alpha \) cannot be too close to 1. Simulation experience indicates that \( \alpha \) is normally within the range \([0.99, 0.999999]\). The effects of \( \alpha \) on the design results will be illustrated in Example 2.

Finally, the design steps of the proposed iterative algorithm are summarized below:

Step 1) Given the ideal frequency response \( D(\omega) \), filter orders \( N \) and \( M \), weighting function \( W(\omega) \), set \( k = 0 \), and choose an initial guess \( x^{(0)} \).

Step 2) Set \( k = k+1 \), and compute \( W^{(k-1)}(\omega) \) by (6), \( \bar{A}^{(k-1)} \) by (37) and \( B(\omega) \) by (19). Then utilize \( \bar{A}^{(k-1)} \) to calculate \( \bar{F}^{(k-1)} \) by (14) and \( g^{(k-1)} \) by (15). Finally, solve for \( q^{(k)} \) the SOCP problem (13) with peak constraints (18).

Step 3) Update the coefficients \( x^{(k)} \) by (9). If the stopping condition (16) is satisfied, or \( k \) exceeds the maximum number of iterations, terminate the iterative procedure. Otherwise, go to Step 2 and continue.

IV. SIMULATIONS

In this section, four examples are presented to demonstrate the effectiveness of the proposed method. We have implemented the proposed design method by SeDuMi [27] in MATLAB. The peak and \( L_2 \) errors of magnitude (MAG) and group delay (GD) are adopted as measurements within frequency bands of interest. In all designs, the step size \( \gamma \) in (9), the parameter \( c \) in (16), and the maximum number of iterations are always set as 0.8, \( 10^{-6} \), and 200, respectively.

A. Example 1

The first example taken from [11] is to design a lowpass digital filter. The ideal frequency response is defined as

\[ D(\omega) = \begin{cases} e^{-j15\omega} & 0 \leq \omega \leq 0.5\pi \\ e^{-186.6(0.5-\omega^2)} & 0.5\pi < \omega < \pi \end{cases} \]

Both numerator order \( N \) and denominator order \( M \) are chosen as 18. The prescribed maximal pole radius is set to \( r = 0.99 \). The weighting function used in this example is set to 1 over the entire frequency band. The parameter \( \alpha \) used in (37) is set to 0.999. All initial numerator coefficients are chosen as 1, and the initial denominator coefficient vector is set to \([1, 0, 0, \ldots, 0]^T\), i.e., all the initial poles locate on the origin of the \( z \)-plane. It takes the algorithm 49 iterations to converge to the final solution. The maximal pole radius of the obtained IIR filter is 0.9226. The magnitude and group delay responses are shown in Fig. 1. For comparison, we also design the lowpass filter using the least 4-power method of [11] under the same specifications. The maximal pole radius of the IIR filter obtained by [11] is 0.9407. The magnitude and group delay responses in the passband are also presented in Fig. 1. The peak and \( L_2 \) errors in the passband are summarized in Table I. It can be seen that the proposed method can achieve better performances in the least squares sense.

In order to illustrate the effectiveness of peak constraints formulated in (18), the ideal frequency response \( D(\omega) \) is then modified as

\[ D(\omega) = \begin{cases} e^{-j15\omega} & 0 \leq \omega \leq 0.5\pi \\ 0 & 0.5\pi \leq \omega < \pi \end{cases} \]

The parameter \( \alpha \) is set as 0.99996. Then, we impose peak constraints on the stopband \([0.55\pi, \pi]\) with \( \mu_i = 0.0178 \) (-35 dB) for \( \omega_i \in [0.55\pi, \pi] \) \( i = 1, 2, \ldots, 90 \). The weighting function is set to 1 over passband and stopband, and 0 over transition band. After 65 iterations, the iterative procedure converges to the final solution. The design results are shown in Fig. 2, and the peak and \( L_2 \) errors are shown in Table II. The maximal pole radius of the obtained filter is 0.9732. We also adopt the WLS method of [9] to design the filter under the same specifications.
Note that the WLS method in [9] is essentially a special case of the least $p$-power method with $p = 2$. The maximal pole radius of the filter designed by [9] is 0.9620. The design results are also shown in Fig. 2, and all measurements are summarized in Table II.

In [9] and [11], the positive-realness based stability constraint is employed to guarantee the stability of designed filters, which is expressed as

$$\Re\{Q(e^{i\omega})\} = q^T \cdot \Re\{\phi_z(e^{i\omega})\} > 0, \quad \forall \omega \in [0, \pi]$$ \hspace{2cm} \text{(39)}

This stability criterion is sufficient. However, it cannot be used to specify an arbitrary maximal pole radius $r$. Results show that the filters designed by the proposed method in this paper does not always satisfy the stability constraint (39), whereas the filter is still stable. 

**B. Example 2**

The second example is to design a halfband highpass filter [15]. The ideal frequency response is given by

$$D(\omega) = \begin{cases} e^{-i2\omega} & \text{if } 0.525\pi \leq \omega < \pi \\ 0 & \text{else} \end{cases}$$

The filter orders are chosen as $M = N = 14$. The prescribed maximal pole radius is set to $r = 1$. The weighting function is chosen as $W(\omega) = 1$ for $\omega \in [0, 0.475\pi] \cup [0.525\pi, \pi]$, and 0 for $\omega \in (0.475\pi, 0.525\pi)$. The parameter $\alpha$ is specified as 0.99996. The initial numerator coefficients are all set to 1 as in Example 1. The initial poles are uniformly located on the unit circle, i.e.,

$$Q^{(0)}(z) = \prod_{i=1}^{M/2} \left[ 1 - z^{-1} e^{-j(\frac{2\pi i}{M})} \right] \left[ 1 - z^{-1} e^{-j(\frac{2\pi (M-i)}{M})} \right]$$

$$= \prod_{i=1}^{M/2} \left[ 1 - 2z^{-1} \cos \left( \frac{2\pi i}{M} - \frac{\pi}{M} \right) + z^{-2} \right]$$ \hspace{2cm} \text{(40)}

It takes the algorithm 72 iterations to converge to the final solution. The design results are shown in Fig. 3. All the peak and $L_2$ errors are summarized in Table III. The maximal pole radius of the designed IIR highpass filter is 0.9782. We also adopt the WISE method of [15] to design the IIR highpass filter under the same specifications. The maximal pole radius of the obtained filter is 0.9852. The design results and all measurements are also given in Fig. 3 and Table III. Apparently, the proposed method can achieve better performances.

In order to demonstrate the effects of the parameter $\alpha$ on the design results, we design the highpass filters with different $\alpha$. We perform the design procedures for 20 times by increasing $\alpha$ from 0.99 to 1. Fig. 4 shows the variation of maximal pole radii with respect to $\alpha$. As mentioned earlier, the poles approach the prescribed maximal pole radius when gradually augmenting $\alpha$. Fig. 5 shows the variation of total number of iterations with respect to $\alpha$. Note that as $\alpha = 1$, the design problem is essentially formulated in the WLS sense without any stability constraint, and the iterative procedure cannot converge within the specified maximum number of iterations. We find that when $\alpha \leq 0.99998$, $A^{(k)}$ in (37) is still positive semi-definite. Fig. 5 shows that we can improve the convergence speed of the iterative procedure by decreasing $\alpha$. However, the maximal pole radius of the designed filter can also be reduced.

**Table I**

<table>
<thead>
<tr>
<th>Method</th>
<th>Passband MAG (Peak/ $L_2$ in dB)</th>
<th>Passband GD (Peak/ $L_2$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Proposed</td>
<td>-18.829/ -37.594</td>
<td>2.754/ 2.691e-1</td>
</tr>
</tbody>
</table>

**Table II**

<table>
<thead>
<tr>
<th>Method</th>
<th>Passband MAG (Peak/ $L_2$ in dB)</th>
<th>Passband GD (Peak/ $L_2$)</th>
<th>Stopband MAG (Peak/ $L_2$ in dB)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Proposed</td>
<td>-29.668/ -47.348</td>
<td>3.675/ 2.221e-1</td>
<td>-35.001/ -47.305</td>
</tr>
</tbody>
</table>

Fig. 1. Design results of Example 1. Solid curves: designed by the proposed method. Dashed curves: designed by the least 4-power method of [11].

Fig. 2. Design results of Example 1 with peak constraints. Solid curves: designed by the proposed method. Dashed curves: designed by the WLS method of [9].

Fig. 3. Design results of Example 1. Solid curves: designed by the proposed method. Dashed curves: designed by the WLS method of [9].
the parameter $\alpha$ should be appropriately selected, such that the maximal pole radius of the designed filter can best approach the specified maximal pole radius $r$. The simulation results in Figs. 4 and 5 suggest a way to choose $\alpha$. First of all, specify $r$, and then set $\alpha = 1$ and perform the design procedure. If the iterative procedure converges and the obtained poles lie inside $C$, the design result can be accepted as the final solution. Otherwise, $\alpha$ should be decreased until a satisfied design result is obtained.

C. Example 3

In this example, another lowpass digital filter with the following ideal frequency response is designed:

$$D(\omega) = \begin{cases} e^{-j12\omega} & 0 \leq \omega \leq 0.4\pi \\ 0 & 0.56\pi \leq \omega < \pi \end{cases}$$

The design specifications are exactly the same as those proposed for the first example in [14]. The filter orders are chosen as $N = 15$ and $M = 4$. The prescribed maximal pole radius is set to $r = 0.84$. Note that the weighting function used in [14] is essentially equivalent to $W^2(\omega)$ used in our formulation. Therefore, we choose the weighting function as

$$W(\omega) = \begin{cases} 1 & 0 \leq \omega \leq 0.4\pi \\ \sqrt{2.6} & 0.56\pi \leq \omega < \pi \\ 0 & \text{otherwise} \end{cases}$$

The parameter $\alpha$ used in (37) is set to 0.999992. The initial coefficients are chosen as $p^{(0)} = [1, 1, \ldots, 1]^T$, and $q^{(0)} = [1, 0, 0, \ldots, 0]^T$. After 12 iterations, the algorithm finds the final solution. The design results are presented in Fig. 6. All the peak and $L_2$ errors are shown in Table IV. The maximal pole radius of the designed IIR digital filter is 0.7986. We also utilized the WLS method of [14] to design the filter. The maximal pole radius of the obtained filter is 0.7233. The design results and all measurements are also given in Fig. 6 and Table IV.

In [14], a stability constraint based on the linearized argument principle (AP) is incorporated in the iterative design progress. At the 4th iteration, the stability constraint (32) is replaced by its linear approximation, i.e.,

$$\tau(r, q^{(k)}) = \tau(r, q^{(k-1)}) + \nabla^T \tau(r, q^{(k-1)}) \eta^{(k)} = 0$$

Assuming that at the previous iteration all poles lie inside $C$, then we have $\tau(r, q^{(k-1)}) = 0$. Thus, the stability constraint (41) is simplified as

$$\nabla^T \tau(r, q^{(k-1)}) \eta^{(k)} = 0$$

### Table III

<table>
<thead>
<tr>
<th>Method</th>
<th>Passband MAG (Peak/ $L_2$ in dB)</th>
<th>Passband GD (Peak/ $L_2$)</th>
<th>Stopband MAG (Peak/ $L_2$ in dB)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Proposed</td>
<td>-27.769/ -47.505</td>
<td>1.887/ 1.234e-1</td>
<td>-23.064/ -42.801</td>
</tr>
</tbody>
</table>

Fig. 3. Design results of Example 2. Solid curves: designed by the proposed method. Dashed curves: designed by the WISE method of [15].

Fig. 4. Variation of maximal pole radii with respect to $\alpha$.

Fig. 5. Variation of total number of iterations with respect to $\alpha$. 

TABLE III

MEASUREMENTS OF DESIGN RESULTS IN EXAMPLE 2
which is a linear equality constraint with respect to $q^{(k)}$. Like the Rouché’s theorem based stability constraint, the iterative procedure incorporating (42) as the stability constraint has to start from a denominator whose poles lie inside C. Fig. 7 shows the values of $\nabla \tau(r, q^{(k)})\eta^{(k)}$ during the iterative procedure of the proposed method. It can be seen that the maximal pole radius of the designed IIR filter can be still less than $r$ even when the stability constraint (42) is not satisfied.

D. Example 4

The last example is to implement an equalization and anti-aliasing filter [13], [28], which follows an analog anti-aliasing filter and a sampler, to equalize the magnitude and phase (or group delay) responses of the analog filter in the passband and increase the attenuation in the stopband. The ideal frequency response of the cascaded system is defined by

$$D_c(\omega) = e^{-j\omega\tau} \begin{cases} 1 & 0 \leq \omega \leq \frac{\pi}{16} \\ 0 & \frac{3\pi}{16} \leq \omega < \pi \end{cases}$$

Then, the desired frequency response of the IIR digital filter is $D_c(\omega)/H_s(j\omega/T)$, where $H_s(j\omega/T)$ is the frequency response of the analog filter and $T$ denotes the sampling period. The transfer function $H_s(s)$ of the analog filter is given in [28]. The desired delay $\tau_c$ can be used as a free parameter to minimize $\delta$. In [28], the best FIR filter design according to the complex Chebyshev criterion has been presented with $\tau_c = 35$, while an IIR filter with $\tau_c = 32$ has been designed in [13] under the least-squares sense. In our designs, the best result can be obtained when the desired group delay in the passband is set as $\tau_c = 34$. Filter orders are $N = 20$ and $M = 4$. The prescribed maximal pole radius is chosen as $r = 0.99$. The parameter $a$ is selected as 0.99994. The weighting function is set as

$$W(\omega) = \begin{cases} 10 & 0 \leq \omega \leq \frac{\pi}{16} \\ 1 & \frac{3\pi}{16} \leq \omega < \pi \\ 0 & \text{otherwise} \end{cases}$$

The initial point is also as $p^{(0)} = [1, 1, \ldots, 1]^T$, and $q^{(0)} = [1, 0, 0, \ldots, 0]^T$. The proposed method takes 69 iterations to reach the final solution. The maximal pole radius of the designed IIR filter is 0.9673. The magnitude responses, phase errors, and group delays of analog filter, designed IIR equalization and anti-aliasing filter, and cascaded system are shown in Fig. 8. We also designed the filter under the same specifications by using the Gauss-Newton method proposed in [13]. The maximal pole radius of the obtained filter is 0.9810. All the peak and $L_2$ errors of magnitude and group delay responses of designed IIR equalization and anti-aliasing filters are summarized in Table V.

In [13], the Rouché’s theorem is employed to develop a stability constraint: Given an initial denominator $Q^{(0)}(z)$ chosen with all its roots inside C, then all denominators $Q^{(k)}(z)$ ($k = 1, 2, \ldots$) have their roots inside C if the denominator updates $\Delta^{(k)}(z)$ satisfies

$$|\Delta^{(k)}(re^{j\omega})| = 7 |\phi^T( re^{j\omega})\eta^{(k)}| < |Q^{(k-1)}(re^{j\omega})|, \quad \forall \omega \in [0, \pi]$$

where $\eta^{(k)} = [\eta_0^{(k)}, \eta_1^{(k)}, \ldots, \eta_M^{(k)}]^T$ ($\eta_0^{(k)} = 0, k = 1, 2, \ldots$). Note that initial denominator polynomials with roots outside C, such as (40), cannot be used by the design processes which adopt (43) as the stability constraint. However, this prerequisite is not

<table>
<thead>
<tr>
<th>Method</th>
<th>Passband MAG (Peak/ $L_2$ in dB)</th>
<th>Passband GD (Peak/ $L_2$ in dB)</th>
<th>Stopband MAG (Peak/ $L_2$ in dB)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Proposed</td>
<td>-33.069/ -52.705</td>
<td>0.223/ 2.382e-2</td>
<td>-37.890/ -57.177</td>
</tr>
<tr>
<td>Linearized AP [14]</td>
<td>-31.547/ -44.707</td>
<td>0.223/ 5.946e-2</td>
<td>-36.990/ -49.421</td>
</tr>
</tbody>
</table>

Fig. 6. Design results of Example 3. Solid curves: designed by the proposed method. Dashed curves: designed by the WLS method with linearized argument principle based stability constraint of [14].

Fig. 7. Values of $\nabla \tau(r, q^{(k)})\eta^{(k)}$ during the iterative procedure of the proposed method.
required by the proposed method in this paper. Like (39), the Rouché’s theorem based stability constraint is only a sufficient condition to ensure stability. Moreover, in order to guarantee the stability, these two constraints must be satisfied at all frequencies \( \omega \in [0, \pi] \). A traditional way to incorporate these constraints is to impose them on a dense frequency grid, which, however, greatly increases the number of constraints. Unlike (39) and (43), the new proposed stability constraint in this paper is realized over the whole frequency band \([0, \pi]\) instead of at each frequency.

V. CONCLUSIONS

In this paper, an IIR digital filter design method with a new argument principle based stability constraint has been presented. As compared with the previous design methods using the reweighting technique, the design problem is formulated as an iterative SOCP problem, which can handle both linear and SOC constraints. Therefore, some more complicated constraints, such as quadratic peak error constraints, can be further incorporated. Another advantage of the design method is that the stability constraint is deduced from the argument principle, which is both sufficient and necessary for stability. Finally, the proposed stability constraint is imposed on the whole frequency band, which greatly facilitates the design progress. In order to incorporate it into the design procedure, the similar reweighting technique is employed, and the stability constraint is then combined with the iterative procedure. If the iterative procedure converges and parameters \( r \) and \( \alpha \) are appropriately selected, the stability constraint can guarantee the stability of the designed filter. When the maximal pole radius \( r \) is given, by adjusting the parameter \( \alpha \), we can also control poles’ locations. The robustness and effectiveness of the proposed approach has been demonstrated in this paper by four numerical examples.

REFERENCES


