ANALYSIS OF 3-D DIGITAL FILTER STRUCTURES

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ABSTRACT
A partitioned matrix representation approach for time and frequency domain representation and analysis of arbitrary 3-D digital filter structures is presented. Due to the relatively small sizes of the resultant partitioned matrices as compared to that without partitioning, efficient time and frequency domain analysis can be obtained.

INTRODUCTION
As a consequence to the development of 3-D digital filter design [1] - [4], we have a need to develop an efficient method for both time and frequency domain analysis of such filters, especially those filters which have high complexity structures. One possible approach is to generalize the matrix representation approach advanced in [5]. However, the resultant dimensions of the matrix of such an approach is equal to N x N for a filter consisting of N nodes. The number N increases as the complexity of a filter structure increases. Consequently, the computer execution time and storage for various analyses could be prohibitive even for a moderate filter order. In practice, the problem could be overcome by using sparse matrix techniques. In this paper, we present an alternative method by which the matrix representation approach could be modified such that efficient time and frequency domain analysis of 3-D arbitrary digital filter structures can be obtained.

TIME AND FREQUENCY DOMAIN REPRESENTATION
For a 3-D digital filter structure, we have three kinds of delay nodes, namely, the delay nodes of the first, second, and third kinds, which correspond, respectively, to the kinds of delay nodes having the least, medium, and largest numbers of delay nodes. For the sake of reducing computer execution time for the frequency domain analysis, the delay nodes of the first kind are numbered first, we follow by numbering the delay nodes of the second kind, and then the delay nodes of the third kind. The numbering of signal nodes is started by numbering the input node first. The remaining signal nodes are numbered according to their sequence of actual computation [6] - [8].

Original Digital Network
A 3-D arbitrary digital filter of N1 delay nodes of the first kind, N2 delay nodes of the second kind, N3 delay nodes of the third kind, and M signal nodes can be represented as

\[
\begin{bmatrix}
I_{11} & I_{12} & I_{13} \\
I_{21} & I_{22} & I_{23} \\
I_{31} & I_{32} & I_{33}
\end{bmatrix}
\begin{bmatrix}
y_1(z_1, z_2, z_3) \\
y_2(z_1, z_2, z_3) \\
y_3(z_1, z_2, z_3)
\end{bmatrix}
= \begin{bmatrix}
y_0(z_1, z_2, z_3) \\
y_1(z_1, z_2, z_3) \\
y_2(z_1, z_2, z_3)
\end{bmatrix}
\]

where \(I_{11}, I_{12}, I_{13}\) and \(I_{21}, I_{22}, I_{23}\) are, respectively, \(N1 \times N1, N2 \times N2, N3 \times N3\), and \(M \times M\) identity matrices; \(y_1, y_{11}, y_{12}, y_{13}\), \(y_2, y_{21}, y_{22}, y_{23}\), \(y_3, y_{31}, y_{32}, y_{33}\); \(y_0, y_1, y_2\), and \(y\) are real matrices of appropriate dimensions; \(y_0(z_1, z_2, z_3), y_1(z_1, z_2, z_3)\), and \(y_2(z_1, z_2, z_3)\) are, respectively, \(N1 \times 1, N2 \times 1\), and \(N3 \times 1\) vectors of signal values at delay nodes of the first, second, and third kinds; \(y_0(z_1, z_2, z_3)\) and \(X_0(z_1, z_2, z_3)\) are, respectively, \(M \times 1\) and \(M \times 1\) vectors of signal values and inputs at signal nodes.

Transpose Digital Network
The transpose of a 3-D arbitrary digital filter can be obtained by taking the matrix transpose of the entire square matrix of (1) as
Based on (1) and (2), various time and frequency domain representations can be obtained.

Unique Time Domain Representation (TDR)

This provides a unique TDR of any 3-D digital filter. It is obtained by taking the inverse z-transform of (1) as

\[ X_d(n, m, t) = \sum_{k=1}^{3} Y_{d_1} k \cdot X_d(n-1, m, t) + \sum_{k=1}^{3} Y_{d_2} k \cdot X_d(n, m, t) + \sum_{k=1}^{3} Y_{d_3} k \cdot X_d(n, m, t-1) + \sum_{k=1}^{3} Y_{d} k \cdot \delta(n, m, t) \]

where

\[ Y_{d_1} = \sum_{k=1}^{3} Y_{d_1} k \cdot Y_{d_2} k \cdot Y_{d_3} k \cdot \delta(n, m, t) \]

State-Space (Non-Unique) Time Domain Representation (TDR)

This provides a state-space but non-unique TDR of a 3-D digital filter with reduced computational complexity. The TDR of an original digital filter can be obtained from (3) - (6), and that of the transpose filter can be obtained from (2).

Original digital network

\[ X_d(n, m, t) = \sum_{k=1}^{3} X_d(n, m, t) \cdot X_d(n-1, m, t) + \sum_{k=1}^{3} X_d(n, m, t) \cdot X_d(n, m, t-1) + \sum_{k=1}^{3} X_d(n, m, t) \cdot \delta(n, m, t) \]

Frequency Domain Representation (FDR)

Upon simplification, the FDR of an original digital filter can be obtained from (7) - (10) and that of the transpose filter can be obtained from (27) - (30).
TIME DOMAIN ANALYSIS

Impulse Response Analysis

By unique TDR

The response at a signal node \( j \), \( y_{cj}(n, m, \ell) \), due to an impulse at a signal node \( i \), \( x_{ci}(n, m, \ell) \), can be obtained from (3) - (6) by setting

\[
x_{ck}(n, m, \ell) = \begin{cases} 1 & 
\end{cases}
\]

(50)

for \( n < 0 \) and/or \( m < 0 \) and/or \( \ell < 0 \)

By state-space (non-unique) TDR

The computational requirement of the impulse response of a 3-D digital filter can be reduced by using (7) - (10) of the state-space TDR instead of (3) - (6) of the unique TDR. Moreover, according to the interreciprocal property of transpose digital networks [5], one single analysis of (27) - (30) gives the total response at a signal node \( j \) due to impulses from arbitrary signal nodes of an original 3-D digital filter.

Granularity and Overflow Limit Cycles Simulation

Simulation of limit cycle effects of a 3-D digital filter due to finite arithmetic operations could be obtained by using the unique TDR given in (3) - (6) and by programming the computer to multiply, add, and operate according to the actual operations of the hardware.

Structural Transformation

The unique TDR could be used as a basis for structural transformation [6], [9] of a 3-D digital filter. This is done by multiplying the delay node signal vectors, \( \text{Y}_{d1}(n, m, \ell), \text{Y}_{d2}(n, m, \ell), \text{Y}_{d3}(n, m, \ell) \), respectively, by real non-singular matrices \( \gamma_1, \gamma_2, \gamma_3 \), of dimensions \( N_1 \times N_1, N_2 \times N_2, N_3 \times N_3 \), respectively.

FREQUENCY DOMAIN ANALYSIS

Frequency Response Analysis

Original digital network

On substituting

\[ z_i = e^{j\omega_i} \]

for \( i = 1, 2, 3 \)

into (31), \( \text{Y}_{d1}(\omega_1, \omega_2, \omega_3) \) due to an input at a signal node \( i \) can be obtained from

\[ \text{Y}_{d1}(\omega_1, \omega_2, \omega_3) = \frac{1}{2} \left[ \text{Re}(\omega_2, \omega_3) + j \text{Im}(\omega_2, \omega_3) \right] \text{Y}_{d}(\omega_1, \omega_2, \omega_3) \]

(52)

where \( \text{Re}(\omega_2, \omega_3) \) and \( \text{Im}(\omega_2, \omega_3) \) represent,
respectively, the real and imaginary parts of \( H(\omega_2, \omega_3) \), and \( f_{lc}(\omega_2, \omega_3) \) represents the \( i \)th column vector of \( F(\omega_2, \omega_3) \). Similarly, \( Y_{d_1}(\omega_1, \omega_2, \omega_3) \) and \( Y_{d_3}(\omega_1, \omega_2, \omega_3) \) due to an input at a signal node \( i \) can be solved, respectively, by direct substitution of (51) into (32) and (33). Finally, the real part, \( Y_{rc_1}(\omega_1, \omega_2, \omega_3) \), and the imaginary part, \( Y_{ic_1}(\omega_1, \omega_2, \omega_3) \), of \( Y_{d_3}(\omega_1, \omega_2, \omega_3) \), which represents the signal value at a signal node \( j \), due to an input at a signal node \( i \) can be obtained from (34) as

\[
Y_{rc_1}(\omega_1, \omega_2, \omega_3) = Y_{d_3}(\omega_2, \omega_3)X_{d_1}(\omega_1, \omega_2, \omega_3) + h_{j_1}(\omega_2, \omega_3)
\]

(53)

where \( X_{d_1}(\omega_2, \omega_3) \) represents the \( j \)th row of \( G(\omega_2, \omega_3) \), and \( h_{j_1}(\omega_2, \omega_3) \) represents the \( j \)th row and \( i \)th column element of \( H(\omega_2, \omega_3) \).

**Transpose digital network**

Using the interreciprocal property of transpose digital networks [5], the total response at a signal node \( j \) due to inputs from all arbitrary signal nodes of an original 3-D digital filter can be obtained by summing the responses at all signal nodes due to an input at the signal node \( j \) of the corresponding transpose digital filter. Hence, on substituting (51) into (45), \( Y_{d_3}(\omega_1, \omega_2, \omega_3) \) due to an input at a signal node \( j \) can be obtained from

\[
Y_{d_3}(\omega_1, \omega_2, \omega_3) = [\cos(\omega_1 - \omega_3) - j \sin(\omega_1 - \omega_3)] C_i Y_{r_1}(\omega_1, \omega_2, \omega_3) + j Y_{i_1}(\omega_1, \omega_2, \omega_3)
\]

(54)

Similarly, \( Y_{d_1}(\omega_1, \omega_2, \omega_3) \) and \( Y_{d_3}(\omega_1, \omega_2, \omega_3) \) due to an input at a signal node \( j \) can be obtained, respectively, from (46) and (47) using (51). Finally, the real part, \( Y_{rc_1}(\omega_1, \omega_2, \omega_3) \), and the imaginary part, \( Y_{ic_1}(\omega_1, \omega_2, \omega_3) \), of the signal value at any signal node \( i \) of the \( u \) signal nodes, \( Y_{d_3}(\omega_1, \omega_2, \omega_3) \), due to an input at a signal node \( j \) can be obtained from (48) as

\[
Y_{rc_1}(\omega_1, \omega_2, \omega_3) = Y_{d_3}(\omega_1, \omega_2, \omega_3)X_{d_1}(\omega_1, \omega_2, \omega_3) + h_{j_1}(\omega_2, \omega_3)
\]

(55)

**Other Frequency Domain Analysis**

Expressions for the first-order coefficient sensitivity, the large-change single-parameter and multi-parameter sensitivity, the group delay and slope of magnitude response analysis, the noise power density spectrum and noise variance analysis of a 3-D arbitrary linear digital filter can be derived in a manner similar to the 1-D and 2-D cases [6] - [8]. Consequently, efficient frequency domain analysis can be obtained using (52), (32), (33), and (53) for an original filter and (54), (46), (47), and (55) for the corresponding transpose filter.

**CONCLUSIONS**

A partitioned matrix representation approach for both time and frequency domain representation and analysis of any 3-D arbitrary digital filter structure has been presented. The present method is flexible and especially useful for high complexity structures where the number of signal nodes over-exceeds the total number of delay nodes. Examples of these structures include 3-D digital filters designed by the wave approach (1-port approach) [1], [2] and the linear transformation approach (2-port approach) [10] - [12].

**REFERENCES**


